FAILURE BY TORSION OF A ROUND ROD WITH A CRACK FOLLOWING ARC OF CIRCLE

N. B. Romalis

The problem of the propagation of a crack taking the form of the arc circle, in torsion of a rod of round cross section, is considered.

An equation is derived for the intensity factor of stresses at the tip of the crack. The local direction of propagation of the crack from a notch perpendicular to the lateral surface of the rod is determined; the approach suggested by Ershov and Ivlev [1] is followed in this work.

1. Consider the propagation of a crack in a round rod of unit radius (Fig. 1).

We know that the characteristic determining the unstable development of a crack is the intensity factor of the stresses at the tip of the crack. The condition for unstable growth of a crack is the Griffith–Irwin condition [2] $k = k_0$, i.e., unstable growth of a crack occurs whenever the stress intensity factor k attains a certain critical value k_0, which is a constant of the material.

According to Sih [3], in torsion

$$k = i\sqrt{2}\tau\mu \frac{f'(\zeta_0)}{\sqrt{\omega''(\zeta_0)}} \tag{1.1}$$

Here τ is the angle of twist per unit length; μ is the shear modulus; $\omega(\zeta_0)$ is the value of the function mapping the region in question onto the circle $|\zeta| \le 1$, at the point ζ_0 corresponding to the tip of the crack $f(\zeta_0)$ is a function of the stresses at the point ζ_0.

Mapping of the region depicted in Fig. 1 onto the circle $|\zeta| \le 1$ is done by the function

$$\omega(\zeta) = \frac{c}{a}\,\frac{1 + 2ai\zeta - \zeta^2 - 2b(1 + i\zeta)\sqrt{2(1-\zeta^2)}}{\zeta^2 + 2ci\zeta - 1}$$

$$a = \frac{(1+\alpha)(1+\bar\alpha)}{3 - (\alpha+\bar\alpha) + 3\alpha\bar\alpha}, \quad b = \frac{1 - \alpha\bar\alpha}{3 - (\alpha+\bar\alpha) + 3\alpha\bar\alpha}, \quad \alpha = \alpha_0 + i\alpha_1, \ \bar\alpha = \alpha_0 - i\alpha_1, \ |\alpha| \le 1 \tag{1.2}$$

$$c = \frac{(1+\bar\alpha)^2}{1 - 6\bar\alpha + \bar\alpha^2}, \quad \alpha = \beta\,\frac{1-\bar\beta}{1-\beta}, \quad \beta = \beta_0 + i\beta_1, \quad \bar\beta = \beta_0 - i\beta_1, \quad |\beta| \le 1$$

The coordinates of the tip of the crack β_0 and β_1 are related to the length s and radius r of the crack by the formulas

$$\beta_0 = 1 - r\sin(s/r), \qquad \beta_1 = r - r\cos(s/r)$$

The tip of the crack $z = \beta$ is mapped into the point $\zeta_0 = -1$. The denominator of the function $\omega(\zeta)$ has the roots

$$\zeta_1 = -i\,\frac{1 - \bar\alpha - 2(\bar\alpha)^{1/2}}{1 - \bar\alpha + 2(\bar\alpha)^{1/2}}, \quad \zeta_2 = -\frac{1}{\zeta_1}, \quad |\zeta_1| \le 1, \quad |\zeta_2| \ge 1$$

The branch of the square root term appearing in Eq. (1.2) is selected such that the numerator of (1.2) vanishes at $\zeta = \zeta_1$. The stress function has the form [4]:

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$$f(\zeta) = -\frac{2bc\bar{c}}{a^2}\frac{i(1+i\zeta)(1+2ai\zeta-\zeta^2)\sqrt{2(1-\zeta^2)}}{(\zeta^2+2ci\zeta-1)(\zeta^2+2\bar{c}i\zeta-1)}$$

$$+\frac{8b^2c\bar{c}}{\pi a^2}\frac{(1+i\zeta)^2(1-\zeta^2)}{(\zeta^2+2ci\zeta-1)(\zeta^2+2\bar{c}i\zeta-1)}\log\frac{1-\zeta}{1+\zeta}+\frac{p}{\zeta-\zeta_1}+\frac{\bar{p}}{\zeta+\bar{\zeta}_1}$$

$$-\frac{p\zeta_2^2}{\zeta-\zeta_2}-\frac{\bar{p}\bar{\zeta}_2^2}{\zeta+\bar{\zeta}_2}+\text{const},\quad \log\frac{1-\zeta}{1+\zeta}=-2\left(\zeta+\frac{1}{3}\zeta^3+\frac{1}{5}\zeta^5+\dots\right) \tag{1.3}$$

$$p=\frac{8b^2c\bar{c}}{\pi a^2}\frac{(1+i\zeta_1)^2(1-\zeta_1^2)}{(\zeta_1-\zeta_2)(\zeta_1+\bar{\zeta}_1)(\zeta_1+\bar{\zeta}_2)}\log\frac{-1-\zeta_1}{1-\zeta_1}$$

For τ in (1.1), we have [5]

$$\tau = M/D, \qquad D = \mu I + \mu D_0 \tag{1.4}$$

Here M is the moment of the applied external forces, D is the rigidity in torsion, I is the polar moment of inertia of the cross sectional area of the rod with respect to the center, and D_0 is defined by the equation (here γ is the circumference of the circle $|\zeta| \leq 1$)

$$D_0 = -\frac{1}{4}\int_\gamma \{f(\sigma)+\overline{f(\sigma)}\}\, d\{\omega(\sigma)\overline{\omega(\sigma)}\}$$

$$= -\frac{1}{4}\int_\gamma\Bigg\{-\frac{2b^2c\bar{c}}{a^2}\frac{(1+i\sigma)(1+2ai\sigma-\sigma^2)}{(\sigma^2+2ci\sigma-1)(\sigma^2+2\bar{c}i\sigma-1)}\left(\sqrt{2(\sigma^2-1)}+i\sqrt{2(1-\sigma^2)}\right)$$

$$+\frac{16b^2c\bar{c}}{\pi a^2}\frac{(1+i\sigma)^2(1-\sigma^2)}{(\sigma^2+2ci\sigma-1)(\sigma^2+2\bar{c}i\sigma-1)}\log\frac{1-\sigma}{1+\sigma}$$

$$+\frac{2p}{\sigma-\zeta_1}+\frac{2\bar{p}}{\sigma+\bar{\zeta}_1}-\frac{2p\zeta_2^2}{\sigma-\zeta_2}-\frac{2\bar{p}\bar{\zeta}_2^2}{\sigma+\bar{\zeta}_2}-2p\zeta_2+2\bar{p}\bar{\zeta}_2\Bigg\}$$

$$\times d\Bigg\{\frac{c\bar{c}}{a^2}\frac{(1+2ai\sigma-\sigma^2)^2}{(\sigma^2+2ci\sigma-1)(\sigma^2+2\bar{c}i\sigma-1)}+\frac{4b^2c\bar{c}i(1+i\sigma)^2\sqrt{2(1-\sigma^2)}\sqrt{2(\sigma^2-1)}}{a^2(\sigma^2+2ci\sigma-1)(\sigma^2+2\bar{c}i\sigma-1)}$$

$$-\frac{2bc\bar{c}}{a^2}\frac{(1+i\sigma)(1+2ai\sigma-\sigma^2)}{(\sigma^2+2ci\sigma-1)(\sigma^2+2\bar{c}i\sigma-1)}\left(\sqrt{2(1-\sigma^2)}+i\sqrt{2(\sigma^2-1)}\right)\Bigg\}$$

In the integrand, the points $\sigma = \pm 1$ are the branch points of the functions $\sqrt{2(\sigma^2-1)}$ and $\sqrt{2(1-\sigma^2)}$. We represent the integral taken over the contour γ in the form of a sum of three integrals

$$D_0 = -\frac{1}{4}\left[\int_\gamma \{\varphi_1(\sigma)\sqrt{2(1-\sigma^2)}+\varphi_0(\sigma)\}\,d\sigma+\int_\gamma \varphi_2(\sigma)\sqrt{2(\sigma^2-1)}\,d\sigma+\int_\gamma \varphi_3(\sigma)\sqrt{2(1-\sigma^2)}\sqrt{2(\sigma^2-1)}\,d\sigma\right]$$

The integrand function in the first integral is single-valued within the circle (Fig. 2), where the branch points $\sigma = \pm 1$ are excluded from the region. Arcs γ_1 and γ_2 are infinitesimally small. The first integral is calculated by applying the theorem on residues. The residues are taken at the point $\sigma = \zeta_1$ and $\sigma = -\bar{\zeta}_1$.

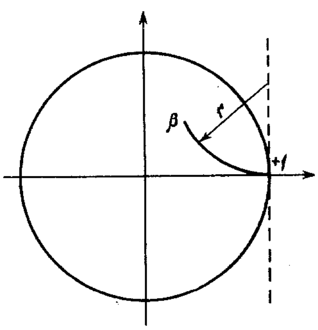

Fig. 1

The integrand function in the second integral is single-valued throughout the interior of the curve ν (Fig. 3). It is calculated in a manner similar to the first. The integrand has poles at the points $\sigma = \zeta_2$ and $\sigma = -\bar{\zeta}_2$.

We now consider the third integral. The integrand function will be single-valued in the region bounded by the contour γ and cut as shown in Fig. 4. The circles γ_1, γ_2, γ_3 are infinitesimally small. The function $\varphi_3(\sigma)$ possesses poles at the points $\sigma = \zeta_1$; we therefore have [5]:

$$\frac{1}{2\pi i}\left[\int_\gamma+\int_\alpha+\int_\beta+\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_3}\right]=A_1+A_2$$

where A_1 and A_2 are the residues taken at the points $\sigma = \zeta_1$ and $\sigma = -\bar{\zeta}_1$.

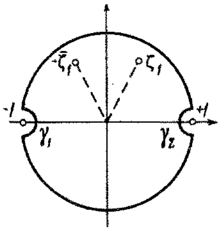

Fig. 2

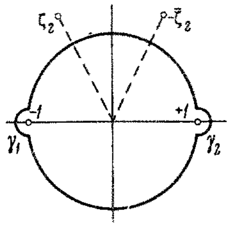

Fig. 3

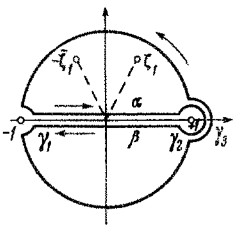

Fig. 4

Carrying out the calculations of all the residues and integrals, we get

$$
\begin{aligned}
\frac{D}{\mu} = \frac{\pi}{2} + \frac{a_1^2}{8\pi\alpha_1^2\sqrt{\alpha_0^2+\alpha_1^2}} \Bigg[& \left\{ \frac{\alpha_1^2 a_1^2}{64(\alpha_0^2+\alpha_1^2)^{3/2}} - \frac{[(1-\alpha_0)^2+\alpha_1^2][(1+\alpha_0)^2+\alpha_1^2]^2}{32 a_2} \right\} (\ln|d|)^2 \\
& + \left\{ -\frac{[(1-\alpha_0)^2+\alpha_1^2][(1+\alpha_0)^2+\alpha_1^2]^2}{8a_0} + \frac{a_3^2}{16(\alpha_0^2+\alpha_1^2)^{3/2}} - \frac{a_1^2\alpha_1^2}{16(\alpha_0^2+\alpha_1^2)^{3/2}} \right\} (\operatorname{arc\,tg} f)^2 \\
& + \frac{\alpha_1 a_1 a_3}{8(\alpha_0^2+\alpha_1^2)^{3/2}} \ln|d| \operatorname{arc\,tg} f + \left\{ -\frac{\sqrt{2}\, a_{10} a_7 [(1-\alpha_0)^2+\alpha_1^2]}{4a_4[(1-\alpha_0)a_6+\alpha_1 a_5]} \right. \\
& + \frac{\sqrt{2}}{8} \frac{[\alpha_1 a_7 a_8 - 8a_1 a_9] a_5 - [a_7 a_9 + 8\alpha_1^2 a_8 a_1] a_6}{a_4(\alpha_0^2+\alpha_1^2)^{3/2}} \\
& \left. - \frac{16\sqrt{2}[\alpha_1(1-\alpha_0)a_5 - \alpha_1^2 a_6] a_1}{a_4} \right\} \ln|d| + \left\{ -32\sqrt{2}\, \frac{a_1[\alpha_1^2 a_5 + \alpha_1(1-\alpha_0)a_6]}{a_4} \right. \\
& - \frac{\sqrt{2}}{2} \frac{[(1-\alpha_0)^2+\alpha_1^2][(1-\alpha_0)^2+4(\alpha_0^2+\alpha_1^2)^{1/2}] a_7}{[(1-\alpha_0)a_5 - \alpha_1 a_6] a_4} \\
& \left. + \frac{\sqrt{2}}{4} \frac{[a_9 a_7 + 8\alpha_1^2 a_1 a_8] a_5 + \alpha_1 [a_8 a_7 - 8a_1 a_9] a_6}{a_4(\alpha_0^2+\alpha_1^2)^{3/2}} \right\} \operatorname{arc\,tg} f - \frac{\alpha_0(1+\alpha_0^2+\alpha_1^2) - 2(\alpha_0^2+\alpha_1^2)}{(\alpha_0^2+\alpha_1^2)^{1/2}} \Bigg]
\end{aligned}
\tag{1.5}
$$

where

$$\alpha_0(1+\alpha_0^2+\alpha_1^2) - 2(\alpha_0^2+\alpha_1^2) + [(1-\alpha_0)^2+\alpha_1^2](\alpha_0^2+\alpha_1^2)^{1/2} = a_0$$

$$1-\alpha_0^2-\alpha_1^2 = a_1$$

$$\alpha_0(1+\alpha_0^2+\alpha_1^2) - 2(\alpha_0^2+\alpha_1^2) - [(1-\alpha_0)^2+\alpha_1^2](\alpha_0^2+\alpha_1^2)^{1/2} = a_2$$

$$(\alpha_0^2+\alpha_1^2)(2+\alpha_0)+\alpha_0 = a_3, \quad (1-6\alpha_0+\alpha_0^2-\alpha_1^2)^2 + (6\alpha_1 - 2\alpha_0\alpha_1)^2 = a_4$$

$$[(\alpha_0^2+\alpha_1^2)^{1/2}+\alpha_0]^{1/2} = a_5, \quad [(\alpha_0^2+\alpha_1^2)^{1/2}-\alpha_0]^{1/2} = a_6$$

$$a_1^2 - 4\alpha_0(1+\alpha_0^2+\alpha_1^2) - 8(\alpha_0^2+\alpha_1^2) - 4\alpha_1^2 = a_7, \qquad 2\alpha_0 + (\alpha_0^2+\alpha_1^2) + (\alpha_0^2+\alpha_1^2)^2 = a_8$$

$$\alpha_0^2 - \alpha_1^2 - (\alpha_0^2+\alpha_1^2)^2 + \alpha_0(\alpha_0^2+\alpha_1^2) a_1 = a_9, \qquad (1-\alpha_0)^2+\alpha_1^2 - 4(\alpha_0^2+\alpha_1^2)^{1/2} = a_{10}$$

$$d = \frac{(1-\alpha_0)^2+\alpha_1^2+4(\alpha_0^2+\alpha_1^2)^{1/2} - 2\sqrt{2}(1-\alpha_0)a_6 - 2\sqrt{2}\alpha_1 a_5}{(1-\alpha_0)^2+\alpha_1^2+4(\alpha_0^2+\alpha_1^2)^{1/2} + 2\sqrt{2}(1-\alpha_0)a_6 + 2\sqrt{2}\alpha_1 a_5}$$

$$f = \frac{4\sqrt{2}[\alpha_1 a_6 - (1-\alpha_0)a_5]}{8(\alpha_0^2+\alpha_1^2)^{1/2} - (\alpha_0-1-\alpha_1)^2 - (\alpha_0-1+\alpha_1)^2}$$

Substituting the values of $f'(\zeta_0)$, $\omega''(\zeta_2)$, and τ into Eq. (1.1), we get

$$k = -\frac{(a_1)^{1/2} M}{2\pi\alpha_1 D} \left\{ \frac{\sqrt{2}(a_1)^{3/2}\alpha_1}{\alpha_0^2+\alpha_1^2} + \frac{\sqrt{2}[b_1 a_5 - \alpha_1 b_2 a_6]}{32(\alpha_0^2+\alpha_1^2)^{3/2}} \ln|d| - \frac{\sqrt{2}}{16} \frac{\alpha_1 b_2 a_5 + b_1 a_6}{(\alpha_0^2+\alpha_1^2)^{3/2}} \operatorname{arc\,tg} f \right\} \tag{1.6}$$

where

$$b_1 = \alpha_0(1+\alpha_0)(1-\alpha_0^2) + \alpha_1^2(1-\alpha_0+\alpha_1^2), \quad b_2 = 1+\alpha_0^2+2\alpha_0^3+\alpha_1^2+2\alpha_1^2\alpha_0$$

Note that in the case where the crack is directed radially

$$(r \to \infty), \quad \alpha_1 = 0, \quad \beta_1 = 0, \quad \alpha_0 = \beta_0 = 1 - s$$

Letting α_1 go to zero in Eq. (1.6) and evaluating the indeterminate form, we obtain, after some transformations, an equation for the intensity factor of the stresses at the apex of a radial crack [3]:

$$k = -\frac{(2-s)^{3/2} s^{1/2} M}{(1-s)^2} \frac{2s + (1-s)^{1/2}(mI_0 + I_1)}{2\pi^2 - [2s^2A^2 + (1-s)(A+B)^2]} \tag{1.7}$$

where

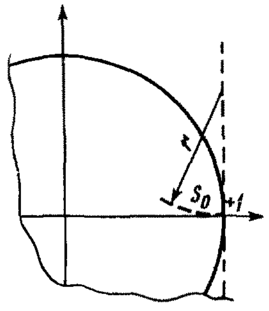

Fig. 5

$$A=\frac{1}{1-s}\left[\frac{(2-s)^2}{(1-s)^{1/2}\operatorname{arc\,tg}(1-s)^{1/2}}-s\right],\qquad B=\frac{s}{1-s}\left[2-\frac{3}{4}As\right]$$

$$m=\frac{(1-s)^2+1}{2(1-s)},\qquad I_0=\frac{4}{\operatorname{arc\,tg}(1-s)^{1/2}},\qquad I_1=-\frac{s}{4(1-s)}[4(1-s)^{1/2-s\cdot}\ I_0]$$

2. The direction in which the crack develops from a notch in brittle failure has been studied [1]. A change in the direction taken by the crack is understood to mean a change in the angle of inclination of the crack to its initial direction.

We now apply the criterion proposed in [1] to the case where a change in the direction of the crack is due to a change in the radius of curvature of the crack.

Let there be a notch of shallow depth s_0 and orthogonal to the lateral surface in a rod of round cross section (Fig. 5). Let δs be the increment in the length of the crack at the initial instant that the crack beings developing away from the notch. We shall assume that the crack is propagating on some arc relative to which the line of the notch is the tangent. Given the smallness of δs, we are free to assume that the crack propagates on an arc of circle of radius r which is to be determined.

In considering the behavior of the stress intensity factor at the tip of the crack, we find the local direction of propagation of the crack characterized by the radius r. The stress intensity factor k depends on s_0, r, and M. At some value of the externally applied load M, the curve $k(s_0, r, M)$, which is treated here as a function of r, comes into contact with the line $k = k_0$ as tangent. The value of M is found from the Griffith–Irwin criterion for a rod with a notch of depth s_0 and is a function of r.

The direction in which the crack develops will be determined by the value of $r = r_0$, which corresponds to the point of tangency $k(s_0, r, M)$ and $k = k_0$. The point r_0 is found from the equation

$$\frac{\partial k(s_0, r, M)}{\partial r}=0 \tag{2.1}$$

Equation (2.1) is the local criterion of the direction of crack growth away from the notch.

Note that the magnitude of the external load shows up in Eq. (2.1) as a multiplier. The direction of crack growth is consequently independent of the load M.

Upon differentiating Eq. (1.6), we calculate the value of

$$\frac{\partial k}{\partial r}=\frac{\partial k}{\partial \alpha_0}\frac{\partial \alpha_0}{\partial r}+\frac{\partial k}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial r}$$

The variables α_0 and α_1 are expressed in terms of the crack length s and crack radius r as follows:

$$\alpha_0=\{1-2r^2+6r^2\cos(s/r)+6r\sin(s/r)-4[r^2\cos^2(s/r)+\sin^2(s/r)+r\sin(2^s/_r)]\}^{1/2}$$

$$\alpha_1=2\sin(s/r)-2r+2r\cos(s/r)$$

Recalling that the notch depth s_0 is shallow, we may assume

$$\sin(s_0/r)=s_0/r,\qquad \cos(s_0/r)=1$$

Then

$$\alpha_0=\left(1-2s_0-\frac{4s_0^2}{r^2}\right)^{1/2},\qquad \alpha_1=\frac{2s}{r}$$

Calculations carried out on the "Promin'" digital computer showed that $\partial k/\partial n = 0$ only in the limit as $r \to \infty$.

This means that the crack will propagate radially. The load parameter corresponding to the onset of crack propagation is found from Eq. (1.6).

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